Assortative Matching with Externalities and Farsighted Agents

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Abstract

We consider a one-to-one assortative matching problem in which matched pairs compete for a prize. With such externalities, the standard solution concept, pairwise stable matching, may not exist. In this paper, we consider farsighted agents and analyze the largest consistent set (LCS) of Chwe (1994). Despite the assortative structure of the problem, LCS tend to be large with the standard effectiveness functions: LCS can be the set of all matchings, including an empty matching with no matched pair. By modifying the effectiveness function motivated by Knuth (1976), LCS becomes a singleton of the positive assortative matching. Our results suggest that the choice of effectiveness function can significantly impact the solution in a matching problem with externalities.

Keywords: group contest, pairwise stable matching, assortative matching, farsightedness, largest consistent set, effectiveness function

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1 Introduction

There is a large literature on two-sided matching problems after a celebrated paper by Gale and Shapley (1962). The structures and the properties of its central solution concept, pairwise stable matching, have been investigated extensively. At the same time, relatively little attention has been paid to matching problems with externalities, despite their ubiquity in many matching markets in the real world. For instance, matched pairs compete after a matching is formed. In this case, Consider the following example. Suppose that there are three male and three female skaters with high, medium, and low ability. It is natural to predict a positive assortative matching as an outcome of this example. Is it pairwise stable under the above effectiveness function? Consider a deviation by the high ability male and the medium ability female skaters from the assortative matching. Then, according to the effectiveness function, their former partners, the high ability female and the medium ability male, cannot participate in the pairs competition, since they become singles. This means that there are only two pairs in the competition, and the deviating pair gets a high winning probability against the low ability pair. Thus, in the presence of externalities, there may not be a pairwise stable matching under the standard effectiveness function.

When the high ability male and the medium ability female agents deviate, they do not expect any reaction from their former partners. Since single agents cannot participate in the pairs contest, it is beneficial to match with any available partner. Given the two singles dumped by their partners are available to form a pair, it is probably not reasonable for the deviating pair to expect their deviation to decrease the number of pairs. Thus, it is natural to investigate whether or not agents'

1.1 Literature Review

stable set is a singleton set of a stable matching under coalitional and pairwise effectiveness functions, respectively. Kimya (2021) showed that the largest maximal farsighted set in the spirit of Dutta and Vartiainen (2020) coincides with LCS by Chwe (1994) in this domain with coalitional deviations.[6] We consider farsighted agents in the pairs competition model with externalities in this paper, and show that the choice of effectiveness function matters, providing an example where LCS, under the standard effectiveness function, is the set of all matchings, including a fully unmatched matching.

Third, our paper belongs to the literature of assortative matching. Becker (1973) introduced the assortative model of marriages. Banerjee, Konishi, and Sönmez (2001) extended Becker's assortative matching problem to hedonic coalition formation problems without externalities by defining a top coalition property. This property guarantees the existence and uniqueness of the core.[7] Diamantoudi and Xue (2003) proved that under the top coalition property, LCS coincides with a singleton core under the standard effectiveness function in coalition formation problems. Mauleon, Vannetelbosch, and Vergote (2011) derived the same result in the context of one-to-one matching. Although our model has the same assortative structure, the results are quite different with externalities.

2 The Model

We first define our one-to-one matching problem with externalities, and introduce basic terminologies in the next subsection, then we move on to introduce (figure skating) pairs competition problem.

[6] Dutta and Vartiainen (2020) introduced history dependence to the rational expectations farsighted stability in Dutta and Vohra (2017) to assure nonemptiness of solutions for all finite problems.

[7] See Bogomolnaia and Jackson (2002) and Leo et al. (2021) as well.

2.1 One-to-One Matching Problems with Externalities

Let $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ be the sets of male and female agents with $|M| = |W| = n$. Let $\mu : M \cup W \to M \cup W$ be a one-to-one matching: $\mu(\mu(x)) = x$ for all $x \in M \cup W$ such that if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. The set of all matchings is denoted by $\mathcal{M}$. Each agent $x \in M \cup W$ has a complete, transitive, and reflexive preference relation R_x which is a binary relation over $\mathcal{M}$. Let the associated strict preference relation be $\mu P_x \mu' \Leftrightarrow (\mu R_x \mu'$ and $\neg \mu' R_x \mu)$, and associated indifference relationship be $\mu I_x \mu' \Leftrightarrow (\mu R_x \mu'$ and $\mu' R_x \mu)$. A matching μ is fully matched if $\mu(x) \neq x$ for all $x \in M \cup W$. Denote a set of all fully matched matchings by $\mathcal{M}^*$. A matching μ is a fully unmatched matching if $\mu(x) = x$ for all $x \in M \cup W$.

We define an effectiveness function which describes the resulting matching induced by a deviation from the original matching. The following effectiveness function is standard in the literature of matching theory and coalition formation (Roth and Vande Vate, 1990; Diamantoudi and Xue, 2003; Herings, Mauleon, and Vannetelboch, 2020).

Definition 1. A matching μ' is induced from μ by a pair $(m, w) \in M \times W$, denoted by $\mu \to_{(m,w)} \mu'$, if it holds

(i) $\mu(m) \neq w$ and $\mu'(m) = w$;

(ii) $\mu(m) \neq m \Rightarrow \mu'(\mu(m)) = \mu(m)$ and $\mu(w) \neq w \Rightarrow \mu'(\mu(w)) = \mu(w)$;

(iii) for all $x \in M \cup W \setminus \{m, w, \mu(m), \mu(w)\}$, $\mu'(x) = \mu(x)$.

In words, the effectiveness function states that when a pair of agents deviates from a matching, the resulting matching is identical to the original matching except that (1) deviators are matched, and (2) their previous partners are single. Similarly, we can define the effectiveness function for a deviation by an agent.

Definition 2. A matching is induced from by an agent $x \in M \cup W$, denoted by $\to$, if it holds

(i) $(x) \neq x$ and $(x) = x$;x

pair i's winning probability is given by

$$= \frac{Y}{\sum_{=1} Y}: \quad (1)$$

The e¤ort cost function is common and linear for every agent x: c (e) = e . Therefore, the expected payo¤s of agent x in pair i is

$$U = e + a_{x( )};$$

where $a_{x( )}$ is agent x's payo¤ from the partner's ability, and > 0 is su¢ ciently small. This is introduced to break ties when there is only one pair in the competition: agent x in the pair prefers a high ability partner even though he/she wins with probability one without making e¤ort. Thus, in the pairs competition problem, for every agent x, preference P satis…es

(i) for all

$n(\ ) \quad jN(\ )j$ is the number of matched pairs under ;

$A(\ ) \quad a^{\frac{1}{}}_{i} + a^{\frac{1}{}}_{x(\ i)} \ ^{\frac{1}{}}$ is the productivity of pair i 2 $N(\ )$.

Pair i's equilibrium winning probability is calculated as[10]

$$= 1 \quad \frac{(n(\ ) \quad 1) \frac{1}{i(x)}}{{}_{=1} \frac{1}{j(x)}}:$$

Member x of pair i's equilibrium payo¤ under when $(x) \neq x$ can be explicitly solved as [11]

$$U = \underbrace{1 \quad \frac{(n(\ ) \quad 1) \frac{1}{i(x)}}{{}_{(x)} \frac{1}{j(x)}}}_{\text{winning probability}} \underbrace{1 \quad \frac{(n(\ ) \quad 1) \frac{1}{i(x)}}{{}_{(x)} \frac{1}{j(x)}} \frac{a}{A(\ )}}_{\text{net bene…ts by taking e¤ort dis utility into account}} + a_{x(\ )}:$$

Since agent x cannot control a , we can write U as:

$$U = V(A(\ ); E(\ )) + a_{x(\ )};$$

where $E(\ )$ is an aggregated externalities index under

$$E(\ ) \quad \frac{{}_{(x)} \frac{1}{j(x)}}{n(\ ) \quad 1}:$$

Note that when agent x gets a higher ability partner, payo¤ U increases due to increases in both $A(\ )$ and $E(\ )$.

It is important to mention two properties of the aggregated externality index $E(\ )$. First, $E(\ )$ tends to decrease in the number of matched pair $n(\ )$. This is because assuming that is the average value of $\frac{1}{j(x)}$, $\frac{1}{(x)} \quad {}_{(x)} \frac{1}{j(x)}$, stays constant, $\frac{(x)}{(x) \quad 1}$ decreases as $n(\ )$ goes up. This externality causes an important di¤erence between the standard matching problem and the one without externalities. The following example mentioned in the introduction illustrates that.

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[10] For the detailed derivations, see Imamura, Konishi, and Pan (2021); Konishi, Pan, and Simeonov (2021).

[11] We can show that if ${}^{n}_{j=1} \frac{1}{A_j(\ )} > (n(\ ) \quad 1) \frac{1}{A_i(\ )}$ for all $i = 1; :::; n$, then every pair gets a positive winning probability, see Imamura, Konishi, Pan (2021); Konishi, Pan, and Simeonov (2021) for the details. This condition is satis…ed for any 2 M if ${}^{n}_{j=1} \frac{1}{}$

Example 1. (Imamura, Konishi, and Pan, 2021) Consider a pairs competition problem with $M = \{m_1; m_2; m_3\}$ and $W = \{w_1; w_2; w_3\}$. Let $a_{\ 1} = a_{\ 1} = 1$, $a_{\ 2} = a_{\ 2} = 0.9$, and $a_{\ 3} = a_{\ 3} = 0.7$. Set $\ = \frac{1}{2}$, then we have $Y\ = (a^{\frac{1}{2}}_{\ i} e^{\frac{1}{2}}_{\ i} + a^{\frac{1}{2}}_{x(\ i)} e^{\frac{1}{2}}_{x(\ i)})^2$ and $A\ = a_{\ i} + a_{x(\ i)}$. For simplicity set $\ = 0$.[12] We calculate m_1's payo¤s under the positive assortative matching and matching with $\ ! \ _1\ _2$.

(i) $\ = \{(m_1; w_1); (m_2; w_2); (m_3; w_3)\}$:

$$U_{\ 1}(\ ) = 1 \quad \frac{2 \quad \frac{1}{2}}{\frac{1}{2} + \frac{1}{18} + \frac{1}{14}} \quad 1 \quad \frac{2 \quad \frac{1}{2}}{\frac{1}{2} + \frac{1}{18} + \frac{1}{14}} \quad \frac{1}{2} = 0.31209$$

(ii) $\ = \{(m_1; w_2); (m_3; w_3)\}$:

$$U_{\ 1}(\ ) = 1 \quad \frac{\frac{1}{19}}{\frac{1}{19} + \frac{1}{14}} \quad 1 \quad \frac{\frac{1}{19}}{\frac{1}{19} + \frac{1}{14}} \quad \frac{1}{1.9} = 0.44720$$

Thus, m_1 is better o¤ by dumping his superior partner for an inferior partner. For any other fully matched matching $\ \in M\ $, a similar deviation blocks $\ $. In addition, for any matching

Lemma 1. (Imamura, Konishi, and Pan, 2021) Let , m ; m 2 M with ` < k (thus a ` > a k), and (m); (m) 2 W with a((m)) < a((m)). Let be such that (m) = (m) and (m) = (m) with (x) = (x) for all other x by swapping the partners among these two pairs. Then, E( ) > E( ) holds.

One important implication of Lemma 1 is that higher ability agents m and (m) are better o¤ by the above assortative swapping, since the abilities of their partners improve. We use these properties to analyze LCS in the next section.

3 The Results

3.1 LCS under the Standard E¤ectiveness Function

In this section, we consider farsighted agents and analyze the largest consistent set (LCS) introduced by Chwe (1994). We begin by providing a few concepts to de…ne LCS.

De…nition 3. A matching is indirectly dominated by if there is a …nite sequence of distinct matchings 0; :::; with 0 = and = such that for every l 2 f0; :::; L 1g; ! +1 holds for some S 2 M [W [M W such that P for x 2 S. We denote this indirect domination by .

De…nition 4. A set of matchings CS(M) M is consistent if for all 2 CS(M), all induced by deviation ! for some S 2 M [W [M W, there exist ~ 2 CS(M) such that ~, and x 2 S with :~P .

De…nition 5. A set of matchings LCS(M) M is the largest consistent set if it is consistent and contains all consistent set C(M) LCS(M).

Denote the positive assortative matching by , where (m) = w for all k = 1; :::; n. In pairs competition problems, satis…es the following property.

Lemma 2. (1) For all $2 M$ with $\neq$, P_1 and P_1 hold, and (2) for all $k = 2; \dots; n$ 1, and all $2 M$ such that (i) $(m) = w$ for all $j = 1; \dots; k$ 1, and (ii) $(m) \neq w$, if $\neq$ then P_k and P_k hold.

Proof. Suppose that $2 M$ and $\neq$. Then, there is k such that $(m) \neq w$. Let the smallest of such k, and name it k. Then, $(m) = w$ holds for all $j = 1; \dots; k$ 1, and $a_{x(\ k)} < a_k$ and $a_{x(\ k)} < a_k$. Consider a deviation by assortative swapping $\rightrightarrows_{(\ k\ k)}$. Since $2 M$, $2 M$ holds. By Lemma 1, we have P_k and P_k , and P_j and P_j for all $j = 1; \dots; k$ 1. Now, suppose that $\neq$. By the same argument, there is the smallest $\ell > k$ with $(m) \neq w$. Consider assortative swapping $\rightrightarrows_{(\ \ell\ \ell)}$, then we have P_k and P_k by Lemma 1. Repeating this argument, we have P_k and P_k o]TJ51.955o15Tf-251(f3)15(e)9. This proves

Due to the assortative structure, one might think that LCS is a singleton of the assortative matching f g. However, the other direction of inclusion relationship does not hold in the model with externalities: LCS includes not only , but also many other matchings. Perhaps surprisingly, LCS in Example 1 coincides with the set of all matchings M, including the empty matching.

Proposition 2. In Example 1, LCS(M) = M.

To prove the above statement, we introduce some notations. Let the sets of matchings with three, two, one, and zero pairs be M^3 = f 2 M : jfx 2 M [W : (x) = xgj = 0g, M^2 = f 2 M : jfx 2 M [W : (x) = xgj = 2g f 2 M : jfx 2 M [W : (x) = xgj g= 2 ability: i.e., for x with (x) 6= x, U ( ) = 1 if (x) = m_1 or (x) = w_1, U = 1 if (x) = m_2 or (x) = w_2, and U = 1 2 if (x) = m_3 or (x) = w_3, where > 0 is arbitrarily close to zero. This construction of payo¤s of single pair matchings guarantees that for all 2 M^1, all 2 M^3 [M^2, and all x with (x) 6= x and (x) 6= x, P holds.

In this particular example, we can also show through direct calculation that for all 2 M^2, all 2 M^3, and all x with (x) 6= x (and (x) 6= x), P holds. The calculations show that U $_1$($_1$) < U $_1$($_2$) holds even though $_1$ is the most preferable matching in M^3 for m_1 and $_2$ is the least preferable in M^2 for m_1. We write down this property formally.

Strong Negative Externalities in Size (SNES). Suppose that (i) 2 M^1 and 2 M^2 [M^3, or (ii) 2 M^2 and 2 M^3. If for all x 2 M [W with (x) 6= x and (x) 6= x, P holds.

With SNES, we can show the following claim.

Claim. In Example 1, M^1 [M^2 [M^3 is consistent.

Proof.

2 M^1, then there is 2 M^2 with by matching a pair excluding the original deviator. Clearly, the original deviator does not bene…t. If a deviation pair creates 2 M^2, then there is 2 M^3 with by matching two single agents. By SNES, the original deviation is not pro…table. If a deviation $(m; w)$ creates 2 M^3 by matching two single agents, then w can deviate with m

3.2 LCS under the Knuth E¤ectiveness Function

We consider the e¤ectiveness function introduced by Knuth (1976) in this section. In our problem, unmatched agents get the lowest payo¤ of zero, since he/she cannot participate in the contest.

Diamantoudi and Xue (2003) showed that if a hedonic game satis…es the top-coalition property, then LCS is the singleton core, which is the assortative matching in the one-to-one matching problem without externalities. Does the same result hold in our problem under the e¤ectiveness function with swapping? The following proposition shows that the answer is a¢ rmative.

Proposition 3. In the pairs competition problem, LCS under e¤ectiveness function $\rightrightarrows$ only includes :i.e., LCS (M) = f g.

Proof. First notice LCS (M) M . If has unmatched singles, any unmatched pair $(m; w)$ can deviate from to obtain positive payo¤s. Since both m and w will have partners under e¤ectiveness function $\rightrightarrows$, after the deviation they retain positive payo¤s, regardless of subsequent deviations. Since m and w obtain zero payo¤s from matching , they certainly deviate from . Thus, 2 LCS (M), and we conclude LCS (M) M .

Now, we will prove LCS (M) = f g. First, we prove the following claim.

Claim. For all 2 LCS (M), we have $(m_1) = w_1$.

Proof of Claim. Consider a set of full matchings in which m_1 and w_1 are not matched: M $_1$ f 2 M : $(m_1) \neq w_1$g. This is a …nite set, and the elements of M $_1$, ^{1}; :::; can be ordered

By the same argument, we conclude $M_2 \cap LCS(M) = \emptyset$. So, we move on to $M_3 \; \{ \in M : (m_1) = w_1; (m_2) = w_2$ and $(m_3) \neq w_3\}$, and so on. This proves that only remains in $LCS(M)$. We completed the proof.□

4 Concluding Remarks

In this paper, we analyzed farsighted agents in a one-to-one matching problem with externalities and the assortative structure. In the matching problem without externalities of Becker (1973), the assortative matching is a quite robust prediction irrespective of pairwise or coalitional deviations and the choice of effectiveness function. However, with externalities, we showed that the choice of effectiveness function is crucial: LCSa29(i)6ities, we wa s8810(n)12912(r)s8811(7)10(3)936(k)39(e)9l(3)9(

solution concept for farsighted agents. The farsighted stable set—vNM stable set defined by indirect domination—have been extensively investigated in the recent literature. It is easy to see that the singleton set of the assortative matching f g is a farsighted stable set in our problem since indirectly dominates any other matchings. The question is whether or not this is the unique farsighted stable set in the pairs competition problem. Harsanyi's (1974) indirect domination requires every coalition participating in the chain reaction of proposals and counter-proposals to

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[12] Chade, H., and J. Eeckhout (2020), Competing Teams. Review of Economic Studies 87.3, 1134-1173.

[13] Chen, B. (2019), Downstream Competition and Upstream Labor Market Matching. International Journal of Game Theory 48.4, 1055-1085.

[14] Fisher, J.C.D., and I.E. Hafalir (2016), Matching with Aggregate Externalities. Mathematical Social Sciences 81 (2016): 1-7.

[15] Gale, D. and L.S. Shapley (1962), College Admissions and the Stability of Marriage, American Mathematical Monthly 69, 9-15.

[16]

[23] Knuth, D.E., (1976), Marriages Stables, Montreal: Les Presses de l'Universite de Montreal.

[24] Kojima, F., and U. Unver, (2008), Random paths to pairwise stability in many-to-many matching problems: a study on market equilibration, International Journal of Game Theory 36 (3), 473-488

[25] Konishi, H., C.-Y. Pan, and D. Simeonov, Team Formation in Contests: Sharing Rules and Tradeo¤s Between Inter- and Intra-Team Inequalities, draft.

[26] Konishi, H., and D. Ray, 2003, Coalition Formation as a Dynamic Process, with Debraj Ray, Journal of Economic Theory, 110, 1-41.

[27] Leo, G., J. Lou, M. Van der Linden, Y. Vorobeychik, M. Wooders (2021), Matching Soulmates, Journal of Public Economic Theory 23 (5), 822-857.

[28] Mauleon, A., V.J. Vannetelbosch, and W. Vergote (2011), Neumann–Morgenstern Farsightedly Stable Sets in Two-Sided Matching, Theoretical Economics 6 (3), 499-521

[29] Mumcu, A., and I. Saglam (2010) Stable One-to-One Matchings with Externalities. Mathematical Social Sciences 60.2, 154-159.

[30] Pycia, M., and M.B. Yenmez. "Matching with externalities." University of Zurich, Department of Economics, Working Paper 392 (2021).

[31] Ray, D. (2008), A Game-Theoretic Perspective on Coalition Formation, Oxford University Press, Oxford.

[32] Ray, D., and R. Vohra (2014), "Coalition Formation," Handbook of Game Theory vol. 4, pp. 239-326.

[33] Ray, D., and R. Vohra (2019), "Maximality in the Farsighted Stable Set," Econometrica 87 (5), 1763-1779.

[34] Rosenthal, R.W., (1972), Cooperative games in e¤ectiveness form, Journal of Economic Theory 5, 88-101.

[35] Sasaki, H., and M. Toda (1996), Two-sided Matching Problems with Externalities, Journal of Economic Theory 70.1, 93-108.

[36] Xue, L., (1998), Coalitional Stability under Perfect Foresight, Economic Theory 11 (3), 603-627.