# Endogenous Alliances in Survival Contests*

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**Abstract**

Esteban and Sakovics (2003) showed in their three-person game that an

the members' rents, even if the alliance wins the first race. Because of this *rent-dissipation* effect, the members of the alliance have lower valuations for winning in the first race, reducing their efforts and the winning probability. Second, even without the rent-dissipation problem, e.g., if the winning prize is shared equally, there are still *free-riding* incentives for the alliance members to reduce efforts and, consequently, the winning probability. As a result, they conclude that it is hard to materialize strategic alliances in a Tullock contest model.[1] Konrad (2009) points out that these disincentives are not specific to Tullock contest models — they also appear in first price all-pay auctions.[2]

However, in the real world, forming alliances in competition is ubiquitous — for example, in research and development activities, and nations in conflicts. In this paper, we provide a simple solution for this alliance paradox by using complementarity in efforts in a general but symmetric $N$-person game.[3] To analyze complementarity, we introduce a simple and tractable CES effort aggregator function to translate alliance members' individual efforts into the alliance's joint effort. We assume that each individual member's marginal effort cost is constant in order to limit the benefits of forming an alliance to effort complementarity only.[4] With strong complementarity in efforts, a larger alliance has the effort advantage relative to a smaller one. Although there are aforementioned disincentives, it makes sense to form an alliance as long as the benefits from complementarity exceed the costs. The complementarity parameter in the CES aggregator provides a simple measure of the strength of incentive to form alliances as its value increases from 0 to 1.[5]

[1] Konrad (2004) considers an asymmetric all-pay auction game with exogenously determined hierarchical tournament structure, and shows that the highest valuation player may not have a chance to become the final winner depending on the hierarchical structure.

[2]

We are not the first to present this idea. Following Cornes (1993) and Cornes and Hartley (2007) in the literature of private provision of public goods, Kolmar and Rommeswinkel (2013) and Choi, Chowdhury, and Kim (2016) have already demonstrated the presence of such incentives in alliance formation (see next section). This paper goes one step further. Since players' payoffs are related to the whole alliance structure, it is important to know how other players react to the alliance structure and whether or not the alliance structure could be stable. Therefore, we need to see players' and alliances' strategic interactions, and what happens in equilibrium: in particular, we ask whether or not there exists an equilibrium alliance structure.

We set up a simple alliance (coalition) formation game with multiple stages. In stage 1, players form alliances. In stage 2, alliances compete with each other, and in stage 3, the winning alliance members compete with each other for the indivisible prize. The solution concept is the standard subgame perfect Nash equilibrium. Two things should be noted. First, we model the alliance formation process as an "open-membership" game (Yi 1997) in which players can freely choose their alliance without being excluded.[6] This setup can be motivated by examples of geographical concentrations of specialized retails stores such as car dealers (auto rows). In big cities in the United States, car dealers tend to collocate to form auto rows, despite that they must compete with each other, and that they can choose to stand alone in a different location. Consumers are attracted by auto rows since they can find a wide variety of cars at competitive prices, and stand-alone dealers have a hard time surviving.[7] The prosperity of an auto row depends on the number of retail stores and each store's efforts.[8] Car dealers choose their locations freely, knowing that big auto rows attract many customers, but that the dealers there must face fierce competition with neighboring dealers.[9] Second, given the way we set up the multi-stage game, a singleton-only alliance structure and a grand alliance structure are practically identical, since the former does not have the third stage competition, and the latter does

cost. We can interpret these results that an increase in complementarity within groups intensifies group competition.

[6]In a companion paper, Konishi and Pan (2020), we consider a sequential alliance formation game a la Bloch (1996), and compare the resulting alliance structures (see Conclusion section).

[7]See Konishi (2005) for a mechanism of the emergence of concentration of retail stores.

[8]Note that a Tullock contest success function is identical to consumers' logit demand function in a discrete choice model.

[9]Another possible example is competing technologies that have network externalities: A classic instance is the videotape format war between VHS by JVC and Betamax by Sony in the late 1970s. Japanese electric appliance companies chose one of these two technologies (JVC, Panasonic, and RCA for the former, and Sony, Toshiba, and Sanyo for the latter), but VHS won the market against Betamax. The market competititon took place among the winning technology adopters.

not have the second stage competition. The outcome of these two alliance structures

## 1.1 Literature Review

nously formed groups using a CES effort-aggregator function when group-members *have heterogeneous abilities*. Assuming that the winning prize is enjoyed by all members of a winning team as a public good, they analyze how effort complementarity affects members' efforts. They find that the complementarity parameter has no effect on equilibrium efforts if groups are homogeneous. If groups are heterogeneous, then the divergence of efforts among group-members and, somewhat surprisingly, the winning probability decreases as the complementarity of efforts goes up, contradicting common intuitions that complementarity of efforts solves the free-riding problem.

tion games, including a sequential coalition formation game in Bloch (1996), Okada (1996), and Ray and Vohra (2001). Sanchez-Pages (2007a) explores different types of stability concepts, including sequential coalition formation games in alliance formation in contests where efforts are perfect substitutes. Sanchez-Pages (2007b) considers various stability concepts in a model where players allocate endowment into productive and exploitative activities. These papers assume the award is divisible, and alliance members can write a binding contract of sharing rule in the case of the alliance's winning. In our paper, we do not allow for any side payment, and players cannot credibly commit to any intra-alliance distribution rule as in Esteban and Sakovics (2003). We only focus on the benefits of forming a larger group through complementarity of effort and analyze the endogenous formation of alliances in Tullock contests.

## 2 The Model

There are $N$ players who seek to get an indivisible prize (say, to be the head of an organization). There is no side payment allowed. The set of players is also denoted by $N = \{1, ..., N\}$, and they can form alliances exclusively for the purpose of being the final winner. Each player $i \in N$ can make an effort to enhance the popularity of her alliance and that of herself. We assume that each player has an identical linear cost function $C(e_i) = e_i$ for all $e_i \geq 0$.

Starting from the inter-alliance contest, we introduce potential benefits for players who belong to an alliance — complementarity in aggregating efforts by all alliance members. That is, if player $i$ belongs to alliance $j$ with $N_j \subseteq N$ as the set of members, and these members make efforts $(e_{hj})_{h \in N_j}$, then the aggregated effort of alliance $j$, $E_j$, is described by a CES aggregator function

$$E_j =$$

Stage 1. All players $i$ 2

**Proposition 1.** *Suppose that the winning alliance of the first stage has size $n_j$. Then, the third-stage equilibrium strategy and payoff are*

$$\hat{e}_i = \frac{n_j - 1}{n_j^2} \text{ and } \hat{V}^j = \frac{1}{n_j}\left(1 - \frac{n_j - 1}{n_j}\right) = \frac{1}{n_j^2}.$$

## 3.2 Stage 2: Contest between Alliances

Consider an inter-alliance contest problem. From Proposition 1, we know that for a given size of alliance $n_j$ the payoff of intra-alliance contest is determined by $\hat{V}_j = \frac{1}{n_j^2}$. Thus, the second stage maximization problem of a player $ij$ in alliance $j$ is to maximize the payoff

$$\begin{aligned} V_{ij} &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h\neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h\neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j'\neq j} E_{j'}} \hat{V}_j - e_{ij} \\ &= \frac{\left(e_{ij}^{1-\sigma} + \sum_{h\neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(e_{ij}^{1-\sigma} + \sum_{h\neq i} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + \sum_{j'\neq j} E_{j'}} \frac{1}{n_j^2} - e_{ij}. \end{aligned}$$

The first-order condition with respect to $e_{ij}$ (if an interior solution) is

$$\sum_{j'} E_{j'} - E_j$$

or

Let $X_{-j} = \sum_{j' \neq j} x_{j'}$. Then, $x_j > 0$ is a unique best response to $X$

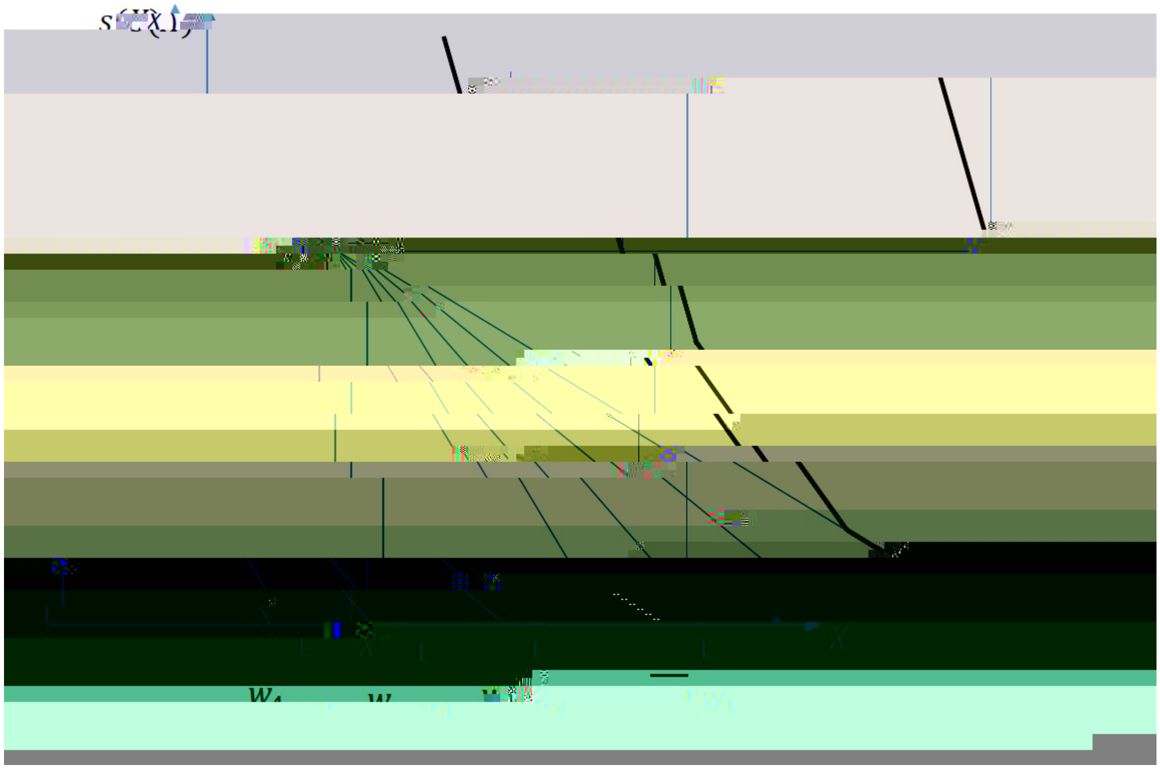

Figure 1

*There is* $j \in \{1, \dots, J\}$ *such that* $p_j = s_j(X\ ) > 0$ *(active alliance) for all* $j \quad j$
$(\hat{X}_j > X\ )$, *while* $p_j = s_j(X$

as  increases, we consider three values of  in order:  $= \frac{1}{2}$ (weak complementarity), $\frac{3}{4}$ (moderate complementarity), and $\frac{4}{5}$ (strong complementarity). We investigate how alliance structure is a ected by the complementarity of team e orts.

### 4.1 Weak Complementarity  $= \frac{1}{2}$

In this case, we have $\frac{2\;\;3\sigma}{1\;\;\sigma} = 1$ and $\frac{1\;\;2\sigma}{1\;\;\sigma} = 0$. Using Theorem 1, we know the following:

| | |
|---|---|
| | $u$ |

### 4.4 Observations

The above examples show that when  is small, there is no gravity to sustain an alliance, since the effort complementarity is not sufficient enough to compensate Olson's inefficiency of alliances.[13] In this case, players prefer standing alone and competing with other single players and/or alliances. In contrast, if  is large, a larger alliance is always relatively more attractive than a smaller alliance, resulting in the grand alliance. When  is in the middle range, nontrivial alliances can appear and Pareto-dominate trivial allocation. For nontrivial equilibria, the complementarity is strong enough to make a singleton player unprofitable. At the same time, it is not strong enough that players prefer a smaller group to avoid severe competition in the final stage. These two forces jointly ensure stability. We will show that this is not a coincidence.

## 5 Two Competing Alliances

We start with the case where the number of (active) alliances is two. We first show a unique two-alliance equilibrium in which the maximal di erence in sizes is one. Denoting $t = \frac{3\sigma \quad 2}{1 \quad \sigma}$, we have the following result.

**Theorem 2.**

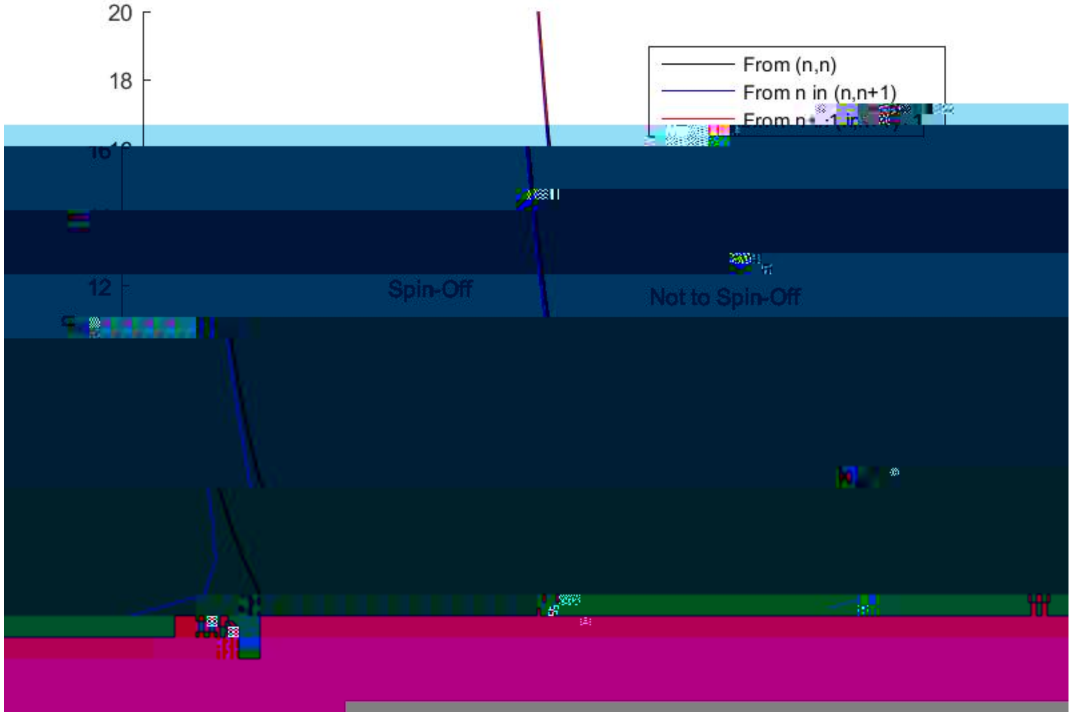


Figure 2: *No Spin-Off Conditions.*

The following theorem shows an important welfare implication of having a chance to form alliances. The emergence of alliances in subgame perfect equilibrium is not only an equilibrium phenomenon (like prisoners' dilemma games), but also a Pareto-improvement for players' welfare, because it has dynamic contests instead of a single round contest.

**Theorem 3.** *Every two-alliance equilibrium* $\{n_1, n_2\}$ *with* $|n_1 - n_2| \leq 1$ *Pareto-dominates a no-alliance contest outcome.*

## 5.1 Multi-Alliance Case

Is a symmetric alliance structure, i.e., all alliances are of the same size, stable when $J > 2$? First of all, forming multiple alliances may be welfare-improving. In fact, if the alliances are symmetric, players' welfare improves as the number of alliances increases. Formally,

**Proposition 3.** *Let symmetric alliance structure $_J$ be a structure that has $\frac{N}{J}$ 2 players in each alliance. If $_{J'}$ and $_{J''}$ with $J'' > J'$ are both equilibrium alliance structures, then $_{J''}$ Pareto dominates $_{J'}$.*

However, the remaining question is whether a multi-alliance structure is stable or not. The benefit from forming a larger alliance is that the new alliance has a higher winning probability in the inter-alliance contest. However, this effect is offset by a stronger intra-alliance competition in the third stage. This winning-probability-enhancement effect is stronger if each alliance only has a smaller number of members and is weaker if the number of alliances is larger. Thus, we expect that, when the number of alliances is more than two, it requires a larger membership in each alliance to be a symmetric equilibrium allocation. This intuition leads us to the following example.

**Example 1.** Consider the case when $J = 3$, $n = 7$ or 8, and $= \frac{3}{4}$

$$u(7;\{7;7;7\}) = 0.0061548 < u(8;\{8;6;7\}) = 0.0061581$$

$$u(8;\{8;8;8\}) = 0.0047743 > u(9;\{9;7;8\}) = 0.0047736$$

The above example shows that even when the complementarity between players is moderate, a symmetric three-alliance structure is not immune to a unilateral move if $n = 7$. But, a larger membership ($n = 8$) again guarantees stability. In fact, $= \frac{3}{4}$ is the borderline case for No Symmetry Breaking when $J = 3$, as will be seen in Corollary 1.

Proposition 2 says that there is no stable two-alliance structure if 2

Note that $u(1; \alpha') > u(n; \alpha)$ holds for all $n \geq 2$ and all $J \geq 4$; i.e., there exist spin-off incentives, and $\alpha$ cannot be a subgame perfect equilibrium outcome. However, when $J = 5$ and $n = 2$, $u(1; \alpha') = \frac{1}{36}$ and $u(2; \alpha) = \frac{1}{4}\frac{1}{25}(5 - \frac{4}{2}) = \frac{3}{100} > \frac{1}{36}$, no player has incentives to spin off and form a singleton alliance. Moreover, since the size of an alliance has no effect when $\gamma = \frac{2}{3}$, the payoff of deviating from a two-player alliance and forming a three-player alliance is $\frac{1}{9}\frac{1}{5^2}\left(5 - \frac{4}{3}\right) < \frac{3}{100}$. Therefore, $\{2; 2; 2; 2; 2\}$ is in fact a stable structure. Since payoffs are continuous in $\gamma$, this example can be extended to those $\gamma$s that are close to but smaller than $\frac{2}{3}$.

Finally, the following proposition assures that for any number of alliances $J \geq 2$, there is a spin-off incentive for every player who belongs to an alliance, if $\gamma$ is small enough.

**Proposition 4.** *Suppose that $\gamma \leq \frac{1}{2}$. Then, from any alliance structure $\alpha$ with a non-singleton alliance, there is a player with an incentive to spin-off to form a singleton alliance.*

Example 1 seems to imply that players have stronger incentives to join a larger group when there are more alliances, and the parameter space for a stable symmetric alliance structure shrinks as the number of alliances increases as a result. In the following section, we analytically confirm this intuition using a heuristic approach that approximates the case with large alliances.

# 6 Symmetric Alliance Structure with Large Population

In the previous section, we analyzed equilibrium conditions by finding the parameter ranges that discourage forming a larger alliance and satisfy No Spin-Off conditions. In this section, we will try to interpret these conditions in the case of a large population, and thus large alliance sizes. We also generalize our analysis by allowing for different continuation games to observe the relevance of continuation payoffs on the equilibrium alliance structure. Consider the following generalization of Stage 3: After team $j$ wins the inter-alliance competition, the winner of the subsequent inter-alliance competition gets a fraction $q$ as a *private* reward. The remaining fraction $(1 - q)$ is the *public* reward enjoyed by all members on the winning team (Esteban and Ray 2001). Note that if $q = 1$, this corresponds to the original setup. If $q = 0$, then there is no Stage 3 competition. If $0 < q < 1$, it is the *mixed* reward case.

We will show that the generalized model above is equivalent to parameterize the expected continuation payo  for team $j$'s victory as $V(n_j) = 1{=}n_j^{\delta}$.

**Lemma 2.** *When the fraction of private reward is $q$ 2 $[0;1]$, the continuation payo is uniquely written as*

$$V(n_j) = \frac{1}{n_j^{\delta}};$$

*where* $= \ \ln \ qn_j{}^2 + (1 \ \ q) \ = \ln n_j$.

That is, if the continuation game is a simple Tullock contest $q = 1$ (private prize), $= 2$ holds. If $= 1$, this means an equal sharing of $V = 1$ without further rent dissipation, and if $1 < \ < 2$, it can be interpreted as a case *partial rent dissipation within the winning alliance*. If $= 0$ or $q = 0$, this is the public reward case. A slight modi cation of Theorem 1 covers all of these cases:[15]

***Theorem 1'.*** *Suppose that, in the winning size $n_j$ alliance the member's subsequent payo is $V(n_j) = \frac{1}{n_j}$. There exists a unique equilibrium in the second stage game for any partition of players $= fn_1;:::;n_Jg$ characterized by the share function $s(X\ ) = 1$ and a unique $j$ $J$ such that players in alliance $j$ $j$ obtain payo*

$$u_j = \frac{1}{n_j^{\delta}} \, 4 \, 1 \quad (J \quad 1) \qquad n^{\delta}$$

2

large populations. With the  rst-order approximation, we can show that $u_j$ does not increase with such a move if and only if $\frac{\sigma}{1 \quad \sigma} \qquad \frac{J}{J \quad 1}$.[9]

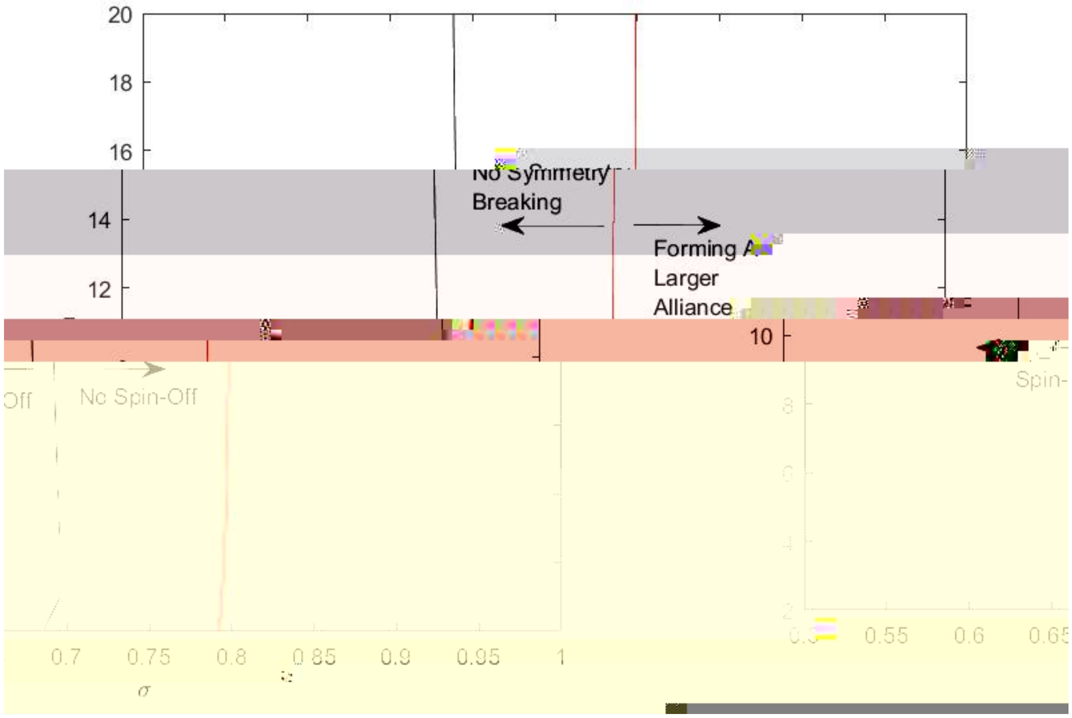
No Symmetry
Breaking
Forming A
Larger
Alliance
No Spin-Off
σ

Figure 3: :

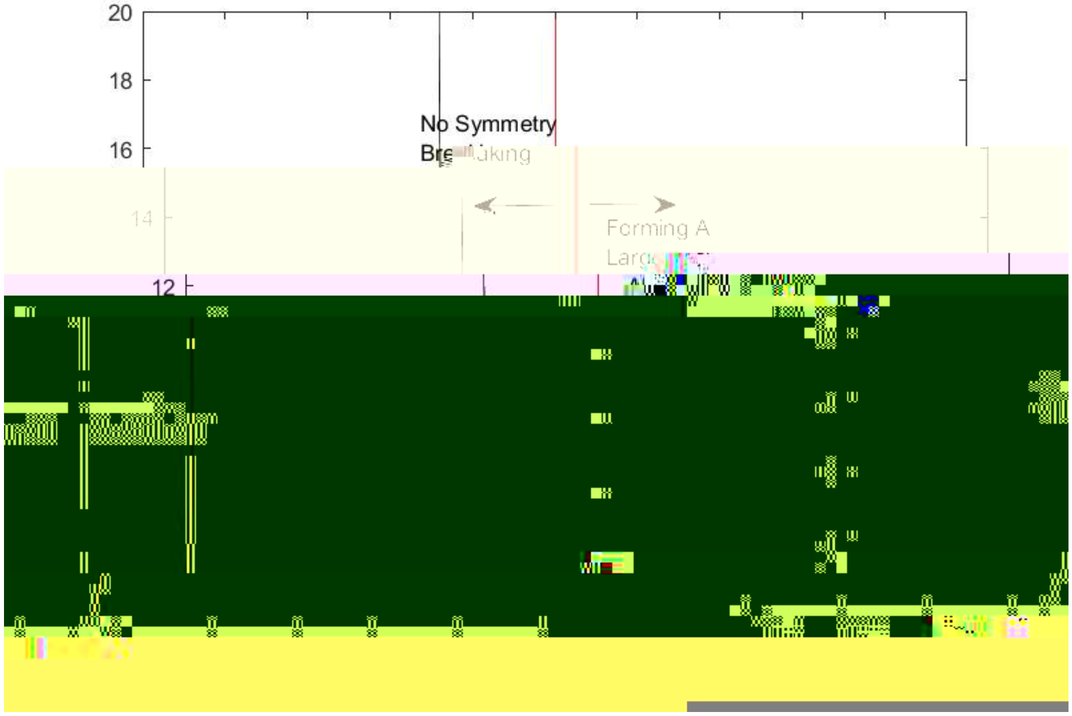


Figure 4: *The stability of a symmetric three-alliance structure.*

If  = 1 and $J$ = 2, then the limit conditions (i) and (ii) in Proposition 5 become $\frac{1}{2}$ $\frac{2}{3}$, so smaller values of achieve stable alliance structures. If the rent dissipation in stage 3 is milder than the simple Tullock contest, such as partial prize sharing (1 < < 2) with $J$ = 2, then the values of for stability are somewhere in between.

For each value of , the values of that support the stability of $J$ symmetric alliance structure are $\frac{\delta}{1+\delta} < \quad < \frac{\delta}{1+\delta \quad \frac{1}{J}}$. Thus, as $J$ goes up, the parameter range of for stable alliance structures shrinks, although players' expected payo s increase.

# 7 Concluding Remarks

In this paper, we used a CES e ort aggregator function to describe incentives to form alliances by e ort complementarity, and we show that there exist stable alliances in an *open-membership* two-stage alliance formation game when the e ort complementarity is moderately strong. When complementarity is too strong, alliances become too attractive, and all players end up forming a grand alliance, which simply defers the noncooperative contest by one period.

There are alternative alliance formation games in the literature (see Hart and Kurz 1983). Using a noncooperative game approach, Bloch (1996), Okada (1996),

[17] Hart, S., and M. Kurz (1983): \Endogenous Formation of Coalitions," *Econometrica* 51(4), 1047-1064.

[18] Herbst, L., K.A. Konrad, and F. Morath (2015): \Endogenous Group Formation in Experimental Contests," *European Economic Review* 74, 163-189.

# Appendix A (Proofs)

**Proof of Theorem 1.** The artificial game we constructed has the same first-order conditions as the original first-stage game. This implies that $j$ is uniquely defined, as in the statement of Lemma 1, only $j = 1, \ldots, j$ exert efforts in equilibrium. Since $p_j = 1 \quad \sum_{j' \neq j}$

Therefore, the equilibrium payo  of the original problem is

$$u_j = p_j \nabla_j \quad e_j$$

$$= \left[ 1 \quad (j \quad 1) \frac{n_j^{\frac{2 \quad 3}{1}}}{\sum_{j'=1}^{j} n_{j'}^{\frac{2 \quad 3}{1}}} \right] \frac{1}{n_j^2} \quad \left[ \frac{1}{n_j^{\frac{1}{1}}} \right] \left[ 1 \quad (j \right.$$

We will compare $u(n_1 + 1; \ {}^{0})$ with $u(n_2; \ )$.

$$u(n_1 + 1; \ {}^{0}) \quad u(n_2; \ )$$

$$= \frac{1}{(n_1+1)^2 + (n_2 \quad 1)^2} \quad \frac{(n_1+1)^2 + (n_2 \quad 1)^2 \quad (n_2 \quad 1)^2 (n_1+1) \quad {}^{1}}{(n_1+1)^2 + (n_2 \quad 1)^2} \; !$$

$$\frac{1}{n_1^2 + n_2^2} \quad \frac{n_1^2 + n_2^2 \quad n_1^2 n_2 \ {}^{1}}{n_1^2 + n_2^2}$$

$$= \frac{1}{(n_1+1)^2 + (n_2 \quad 1)^2} \quad \frac{1}{n_1^2 + n_2^2} \quad \frac{(n_2 \quad 1)^2 (n_1+1) \quad {}^{1}}{(n_1+1)^2 + (n_2 \quad 1)^2 \quad {}^{2}} \; ! + \frac{n_1^2 n_2 \ {}^{1}}{(n_1^2 + n_2^2)^2}$$

$$= \frac{1}{n_2(n_1+1) \quad (n_1+1)^2 + (n_2 \quad 1)^2 \quad {}^{2} \ (n_1^2 + n_2^2)^2}$$

$$n_1^7 \quad 2n_1^6 n_2 + 4n_1^5 n_2^2 \quad 5n_1^4 n_2^3 + 5n_1^3 n_2^4 \quad 4n_1^2 n_2^5 + 2n_1 n_2^6 \quad n_2^7$$

$$+5n_1^6 \quad 12n_1^5 n_2 + 18n_1^4 n_2^2 \quad 20n_1^3 n_2^3 + 17n_1^2 n_2^4 \quad 8n_1 n_2^5 + 4n_2^6$$

$$+12n_1^5 \quad 27n_1^4 n_2 + 28n_1^3 n_2^2 \quad 26n_1^2 n_2^3 + 16n_1 n_2^4 \quad 7n_2^5$$

$$+16n_1^4 \quad 28n_1^3 n_2 + 16n_1^2 n_2^2 \quad 12n_1 n_2^3 + 8n_2^4 + 12n_1^3 \quad 12n_1^2 n_2 \quad 4n_2^3 + 4n_1^2 \ :$$

0, [ ] > 0 holds. Rewriting this, we have

$$
\begin{aligned}
& 4n_2^6 \quad 4n_2^5 + 12n_1^3 \quad 12n_1^2 n_2 \quad 4n_2^3 + 4n_1^2 \\
= \;& 4 \;\; n_2^6 \quad n_2^5 + 3n_1^2 (n_1 \quad n_2) \quad n_2^3 + n_2^2 \quad n_2^2 + n_1^2 \\
= \;& 4 \;\; n_2^6 \quad n_2^5 \quad n_2^3 + n_2^2 + 3n_1^2 (n_1 \quad n_2) + (n_1 \quad n_2)(n_1 + n_2) \\
= \;& 4 \;\; n_2^2 (n_2 \quad 1)^2 \;\; n_2^2 + n_2 + 1 \;\; + 3n_1^2 (n_1 \quad n_2) + (n_1 \quad n_2)(n_1 + n_2) \;\; > 0:
\end{aligned}
$$

n n

We have completed the proof.

: 51

Next, we argue that $u(n_1 + 1; \;^0) \quad u(n_2; \;) > 0$;g127(e)-326(t)-27(v)27(e)]TJ -.7947Td [(4)]TJ

$$\frac{@L}{@t} \quad ' \quad n_1{}^{2t} \ln(n_1+1)\,(n_1+1)^t\,(n_2 \quad 1)^t + n_1{}^{2t} \ln(n_2 \quad 1)\,(n_1+1)^t\,(n_2 \quad 1)^t$$

$$+ n_1{}^t\, n_2{}^t \ln(n_1)\,(n_1+1)^{2t} \quad n_1{}^t\, n_2{}^t \ln(n_2)\,(n_1+1)^{2t}$$

$$+ n_1{}^t\, n_2{}^t \ln(n_1)\,(n_1+1)^t\,(n_2 \quad 1)^t \quad n_1{}^t\, n_2{}^t \ln(n_2)\,(n_1+1)^t\,(n_2 \quad 1)^t$$

$$+ n_1\, n_1{}^t\, n_2{}^t \ln(n_1)\,(n_1+1)^{2t} + n_1\, n_1{}^t\, n_2{}^t \ln(n_1)\,(n_2 \quad 1)^{2t}$$

$$n_1\, n_1{}^t\, n_2{}^t \ln(n_2)\,(n_1+1)^{2t} \quad n_1\, n_1{}^t\, n_2{}^t \ln(n_2)\,(n_2 \quad 1)^{2t}$$

$$n_1{}^t\, n_2{}^t \ln(n_1+1)\,(n_1+1)^t\,(n_2 \quad 1)^t + n_1{}^t\, n_2{}^t \ln(n_2 \quad 1)\,(n_1+1)^t\,(n_2 \quad 1)^t$$

$$+ n_1{}^{2t}\, n_2 \ln(n_1+1)\,(n_1+1)^t\,(n_2 \quad 1)^t \;+ n_1\, n_2{}^{2t} \ln(n_1+1)\,(n_1+1)^t\,(n_2 \quad 1)^t$$

$$n_1{}^{2t}\, n_2 \ln(n_2 \quad 1)\,(n_1+1)^t\,( \quad 1)^t$$

$t$ ( ( ln( 2t ln(n 1-6.73T31F3011.9552Tf4.55022.1665.811T o . Formally,

**Lemma A3.** *For any two-alliance structure* $= (n_1; n_2)$ *with* $n_1 \quad n_2$ $2$*, it is bene cial to spin o from the larger group whenever* $\frac{2}{3}$.

**Proof.** Note that the payo in the size-$n_1$ group is

$$u(n_1; \ ) = \frac{1}{n_1^2} \quad \frac{n_1^t}{n_1^t + n_2^t} \quad \frac{n_1^t + n_2^t \quad n_2^t n_1^{\ 1}}{n_1^t + n_2^t}$$
$$\frac{1}{n_1^2} \quad \frac{n_1^t}{n_1^t + n_2^t} \quad \frac{1}{4} \quad \frac{n_1^t}{n_1^t + n_2^t} \ :$$

Let ${}^{00} = (1; n_1 \quad 1; n_2):$ Then we have

$$u(1; \ ^{00}) = \frac{(n_1 \quad 1)^t + n_2^t \quad (n_1 \quad 1)^t n_2^t}{(n_1 \quad 1)^t + n_2^t + (n_1 \quad 1) n_2^t}^{\ 2} :$$

Note that

$$2 \ \frac{(n_1 \quad 1)^t + n_2^t \quad (n_1 \quad 1)^t n_2^t}{(n_1 \quad 1)^t + n_2^t + (n_1 \quad 1) n_2^t} \qquad \frac{n_1^t}{n_1^t + n_2^t}$$
$$= \frac{n_1{}^t (n_1 \quad 1)^t + 2 n_2{}^t (n_1 \quad 1)^t + 2 n_2{}^{2t} \quad 2 n_2{}^{2t} (n_1 \quad 1)^t + n_1{}^t n_2{}^t \quad 3 n_1{}^t n_2{}^t (n_1 \quad 1)^t}{n_2{}^t + n_2{}^t (n_1 \quad 1)^t + (n_1 \quad 1)^t \quad (n_1{}^t + n_2{}^t)}$$
$$> 0:$$

The last inequality holds because $(n_1 \quad 1)^t \quad 1$ whenever $n_1$

**Lemma A4.** *When $J = 2$, there is $\;\; 2 \; (\frac{3}{4}; \frac{4}{5})$ such that for all $\;\; 2 \; (\frac{2}{3}; \;\; )$, the following statements hold: (i) Players in the smaller alliance do not have an incentive to move to a larger alliance. (ii) When alliance sizes are equal, players do not move to create a larger alliance. (iii) Players in a larger alliance have an incentive to move to the smaller one.*

**Proof of Lemma A4.** Consider $\;\; = (n_1; n_2)$ and $\;\; {}^0 = (n_1 + 1; n_2 \;\; 1)$ with $n_1 \;\; n_2 \;\; 2$. All three statements above are equivalent to

$$\frac{u(n_1 + 1; \;\; {}^0)}{u(n_2; \;\; )} < 1:$$

Suppose there is a $\;\;$ such that at $\;\; = \;\;$, $\frac{u(n_1+1,\pi^0)}{u(n_2,\pi)} < 1$. By Lemma A1 and A2, we know that (a) $\;\; < \frac{4}{5}$ and (b) $\frac{u(n_1+1,\pi^0)}{u(n_2,\pi)} < 1$ holds for all $\;\;$ with $\frac{2}{3} < \;\; < \;\; :$ It remains to show that $\;\; > \frac{3}{4}$. For computational purposes, let $n_1 = n + d + 1$ and $n_2 = n + 1$ with $n \;\; 1$. Note that $n_1 \;\; n_2$ is equivalent to $d \;\; 0$. Consider the case with $\;\; = \frac{3}{4}$. We have

$$u(n_2; \;\; ) = \frac{1}{(n+1)^2} \;\; \frac{n+1}{(n+d+1) + (n+1)} \;\; " \; 1 \;\; \frac{1}{n+1} \frac{n+d+1}{(n+d+1)+(n+1)} \; \#$$

$$= \frac{1}{(n+1)^2} \;\; \frac{n+1}{(n+d+1) + (n+1)} \;\; \frac{(n+d+1) + (n+1) \;\; \frac{n+d+1}{n+1}}{(n+d+1) + (n+1)}$$

$$= \frac{1}{(n+1)^3} \;\; \frac{n+1}{2n+d+2} \;\; n + \frac{n+1}{2n+d+2}$$

$$> \frac{1}{(n+1)^3} \;\; \frac{(n+d+1)^t}{(n+d+1)^t + (n+1)^t} \;\; n + \frac{(n+d+1)^t}{2(n+1)^t}$$

$$= \frac{1}{(n+1)^3} P_0(t) \;\; n + \frac{(n+d+1)^t}{2(n+1)^t} \;\; ; \qquad (4)$$

Now, case (ii). This case is more cumbersome, since a player can spin o from both alliances. We need to consider two possible spin-o subcases $\frac{u(n+1, fn+1, ng)}{u(1, fn, n, 1g)} \quad 1$ and $\frac{u(n, fn+1, ng)}{u(1, fn+1, n \quad 1, 1g)} \quad 1$.

$$! \quad 1n; n$$

$$u(n; fn; n+1g) = \frac{1}{n^2} \quad 1 \quad \frac{\frac{1}{n^t}}{\frac{1}{n^t} + \frac{1}{(n+1)^t}}$$

with Case 1. The payo from $fn;ng$ is $\frac{1}{n^2}\frac{1}{2}$ 1 $\frac{1}{2n}$ = $\frac{2n \quad 1}{4n}$, and the one from $f2ng$ is $\frac{1}{\quad}$

Second, we check $u(n+1;\ )$. We have

$$
\begin{aligned}
u(n+1;\ ) &= \frac{1}{(n+1)^2}\left[1 \quad \frac{(n+1)^{\frac{2\ \ 3}{1}}}{n^{\frac{2\ \ 3}{1}} + (n+1)^{\frac{2\ \ 3}{1}}}\right]\left[1 \quad \frac{(n+1)^{\frac{1\ \ 2}{1}}}{n^{\frac{2\ \ 3}{1}} + (n+1)^{\frac{2\ \ 3}{1}}}\right] \\
& \quad \frac{1}{(n+1)^2}\left[\frac{n^{\frac{2\ \ 3}{1}}}{n^{\frac{2\ \ 3}{1}} + (n+1)^{\frac{2\ \ 3}{1}}}\right]\left[\frac{n \quad 1}{n} + \frac{1}{n}\left(\frac{n^{\frac{2\ \ 3}{1}}}{n^{\frac{2\ \ 3}{1}} + (n+1)^{\frac{2\ \ 3}{1}}}\right)\right] :
\end{aligned}
$$

Since $u(n+1;\ )$ is increasing in for $\frac{2}{}$

$$\frac{\partial u(\pi_J)}{\partial J} = \frac{1}{N^3}(N - 2J + 1).$$

Therefore, $\frac{\partial u(\pi_J)}{\partial J} > 0$ holds for all $J \leq \frac{N+1}{2}$. Also, notice that a group of $N$ players can sustain at most $\frac{N}{2}$ alliances. Therefore, a symmetric structure with more alliances Pareto-dominates one with less.

**Proof of Proposition 4.** From Theorem 1, we know that the payoff of a player who is one of $n_j$ is

$$u(n_j;\pi) = \frac{1}{n_j^2}\left[1 - (J-1)\frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1 - (J-1)\frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right].$$

Let $\pi'_{n_j}$ stand for the structure after one player in alliance $j$ spins off to form a singleton alliance. This player has a payoff equal to

$$u(1;\pi'_{n_j}) = \left[1 - J\frac{1}{\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1}\right]^2.$$

Since $\frac{1-2\sigma}{1-\sigma} \geq 0$, $n_j^{\frac{2-3\sigma}{1-\sigma}} - (n_j-1)^{\frac{2-3\sigma}{1-\sigma}} \geq 1^{\frac{2-3\sigma}{1-\sigma}} = 1$ and $n_j^{\frac{1-2\sigma}{1-\sigma}} \geq 1^{\frac{1-2\sigma}{1-\sigma}} = 1$ hold for all $n_j \geq 2$. Since $n_j^{\frac{2-3\sigma}{1-\sigma}}$ is a convex function for $\sigma \in [0; \frac{1}{2}]$ ($\frac{2-3\sigma}{1-\sigma} \in [1; 2]$), we have

$$\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1 \leq \sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}.$$

This implies

$$\frac{\sum_{j'=1, j'\neq j}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}} + (n_j - 1)^{\frac{2-3\sigma}{1-\sigma}} + 1}{J} < \frac{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}{J-1}.$$

Thus, we have

$$u(1;\pi'_{n_j}) > \left[1 - (J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right]\left[1 - (J-1)\frac{1}{\sum_{j'=1}^{J} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}}\right].$$

We want to show the RHS that the above inequality is not exceeded by $u_j$ for any $2\ [0; \frac{1}{2}]$. Note that $\frac{2\ \ 3\sigma}{1\ \ \sigma}$ 1 and $\frac{1\ \ 2\sigma}{1\ \ \sigma}$ 0 for any changing from $(n_1; n_2)$ to $(n_1 + \quad ; n_2 \quad )$. We have

$$\frac{du_1}{d} = \frac{1}{n_1^{\delta+1}} \left[ 1 \quad \frac{(J \quad 1)n_1^{\delta \; \frac{}{1}}}{D} \right]^2 \left[ 1 \quad \frac{(J \quad 1)n_1^{\delta \; \frac{1}{1}}}{D} \right]$$

$$+ \frac{1}{n_1^{\delta}} \left[ (J \quad 1) \frac{\frac{\sigma}{1 \; \sigma} \; D n_1^{\delta \; \frac{}{1}} \quad 1 \qquad \frac{\sigma}{1 \; \sigma} \; n_1^{\delta \; \frac{}{1}} n_1^{\delta \; \frac{}{1}} \quad 1}{D^2} \right] \left[ 1 \quad \frac{(J \quad 1)n_1^{\delta \; \frac{1}{1}}}{D} \right]$$

$$+ \frac{1}{n_1^{\delta}} \left[ 1 \quad \frac{(J \quad 1)n_1^{\delta \; \frac{}{1}}}{D} \right]^2 \left[ (J \quad 1) \frac{\frac{1}{1 \; \sigma} \; D n_1^{\delta \; \frac{1}{1}} \quad 1 \qquad \frac{\sigma}{1 \; \sigma} \; n_1^{\delta \; \frac{1}{1}} n_1^{\delta \; \frac{}{1}} \quad 1}{D^2} \right]$$

1

$n_1^{\delta} 2$ $\sigma$

Evaluating at $n_1 = n_2 = n_j = n$ and $D = Jn^{\delta \quad \frac{}{1}}$, we obtain,

$$
\begin{aligned}
\frac{du_1}{d} &= \quad \frac{1}{n^{\delta+1}} \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{1}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \\
&\quad \frac{1}{n^{\delta}} \left[ \frac{(J \quad 1)^2 \quad \frac{\sigma}{1 \quad \sigma} \quad n^{2\delta \quad 2\frac{}{1} \quad 1}}{\left(Jn^{\delta \quad \frac{}{1}}\right)^2} \right] \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{1}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \\
&\quad \frac{1}{n^{\delta}} \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \left[ \frac{(J \quad 1)\, n^{2\delta \quad 2\frac{}{1} \quad 2} \; J \quad \frac{1}{1 \quad \sigma} \quad \frac{\sigma}{1 \quad \sigma}}{\left(Jn^{\delta \quad \frac{}{1}}\right)^2} \right] \\
&\quad \frac{1}{n^{\delta}} \left[ (J \quad 1) \frac{\frac{\sigma}{1 \quad \sigma} \quad n^{2\delta \quad 2\frac{}{1} \quad 1}}{\left(Jn^{\delta \quad \frac{}{1}}\right)^2} \right] \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{1}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \\
&\quad \frac{1}{n^{\delta}} \left[ 1 \quad \frac{(J \quad 1)n^{\delta \quad \frac{}{1}}}{Jn^{\delta \quad \frac{}{1}}} \right] \left[ (J \quad 1) \frac{\frac{\sigma}{1 \quad \sigma} \quad n^{2\delta \quad 2\frac{}{1} \quad 2}}{\left(Jn^{\delta \quad \frac{}{1}}\right)^2} \right] \\
&= \frac{1}{n^{\delta+1}} \left[ \frac{}{J} \quad \left(\frac{J \quad 1}{J}\right)^2 \quad \frac{}{1} \quad \frac{J \quad 1}{J^2} \quad \frac{}{1} \quad 1 \quad \frac{J \quad 1}{Jn} \right] \\
&\quad \frac{1}{n^{\delta+2}} \frac{1}{J} \left[ \frac{(J \quad 1) \;\; J \quad \frac{1}{1 \quad \sigma} \quad \frac{\sigma}{1 \quad \sigma}}{J^2} + \frac{(J \quad 1) \quad \frac{\sigma}{1 \quad \sigma}}{J^2} \right] \\
&= \frac{1}{n^{\delta+1}} \frac{1}{J} \quad (J \quad 1) \quad \frac{}{1} \quad 1 \quad \frac{J \quad 1}{Jn} \\
&\quad \frac{1}{n^{\delta+1}} \frac{(J \quad 1)}{nJ^2} \quad \frac{1}{1} \quad :
\end{aligned}
$$

Notice that, if $n$ is large, the second term in the last equation is close to 0. Therefore, when $n$ is large, players have no incentive to move to another alliance unilaterally if

$$\frac{du_1}{d} < 0 \quad (\;) \quad < (J \quad 1) \quad \frac{}{1} \quad :$$

Rearranging the last inequality yields the No Symmetry Breaking condition in Proposition 5.

Now, we turn to the No Spin-O condition. The payo from a sym[redacted]liance structure is simply written as

$$u(n) = \frac{1}{n^{\delta}} \frac{1}{J} \quad 1 \quad \frac{1}{nJ} \; :$$

In contrast, the payo of a player who spun o from a symmetric all[redacted]cture is more subtle, and we need to consider two cases. We start with [redacted]where $\frac{\sigma}{1 \quad \sigma} < 0$. Let $= \frac{\sigma}{1 \quad \sigma}$ If a player spins o[redacted] but we have

$$1 + (J \quad 1) \frac{1}{n^{\epsilon}} + \quad J \; . \; 8$$ [redacted] $[ \; ( rJ$