

ALGEBRA QUALIFYING EXAM SPRING 2018

Exercise 1. Suppose  $p$  is a prime. Show that the Galois group of  $x^5 - 1 \in \mathbb{F}_p[x]$  depends only on  $p \pmod{5}$ , and compute it for each congruence class  $p \pmod{5}$ .

Exercise 2. Let  $R$  be a Dedekind domain with field of fractions  $K$ . Show that for any two proper fractional ideals  $I, J$  there are  $a, b \in K$  with  $aI, bJ \subseteq R$  integral and  $aI + bJ = R$ .

Exercise 3. Suppose that  $R$  is a Noetherian ring and  $\mathfrak{p} \subseteq R$  is a prime ideal such that  $R_{\mathfrak{p}}$  is an integral domain. Show that there is an  $f \in R \setminus \mathfrak{p}$  such that  $R_f$  is an integral domain where  $R_f = S^{-1}R$  with  $S = \{1, f, f^2, \dots\}$ .